variables, and must be treated as variables throughout the mathematical process for finding the maximum value of u.

Differentiating u and equating the differential to zero, we have

$$\frac{Du}{u} = 0 = 2\,(h_1{}^2x_1 + h_2{}^2x_2 + h_3{}^2x_3 + \ldots)\,d\,x$$

$$+ d\,h_1\left(\frac{1}{h_1} - 2\,h_1x^2{}_1\right) + d\,h_2\left(\frac{1}{h_2} - 2\,h_2x_2{}^2\right) + \text{\&c.}$$

$$\therefore\ h_1{}^2x_1{}^2 = h_2{}^2x_2{}^2 = h_3{}^2x_3{}^2, = \text{\&c.} = \tfrac{1}{2}$$

and
$$h_1{}^2x_1 + h_2{}^2x_2 \ldots + h_n{}^2x_n = 0$$

or
$$x = \frac{h_1{}^2a_1 + h_2{}^2a_2 + \ldots}{h_1{}^2 + h_2{}^2 + h_3{}^2 + \ldots}$$

From which we deduce for the determination of x the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n} = 0 \quad (1)$$

The value of x which makes u an absolute maximum must be contained amongst the roots of the equation (1), and this equation will always have one real root which will make u a maximum.

Such is, I believe, the correct solution of the question on the assumptions made, that the weights of each observation are to be deduced entirely from the run of the observations under consideration without any regard to *a priori* probability.

On the Limit of a Possible Lunar Atmosphere. By E. Neison, Esq.

In a preliminary note in the *Monthly Notices* of the Royal Astronomical Society (Vol. xxxiii. p. 464), attention was drawn to the circumstance that no proof of the absence of a sensible lunar atmosphere had as yet been offered, but only that it did not exceed a certain undecided limit. The question possessing no little interest, and being of importance with reference to the present constitution of the Moon's surface, it appeared desirable to fix the limit as satisfactorily as possible within which a lunar atmosphere was possible. For there can be no doubt but that any atmospheric envelope to the Moon, however small in density, must exert a considerable influence on the surface, and enable actions to take place otherwise impossible.

There can be no question but that the main circumstance limiting the density of any possible atmosphere is the refraction of the rays of light it would cause. From what we know on the subject, as there can be no doubt but that it must be of comparatively small density, it is evident we have only to deal with the hori-

zontal refraction, as the refraction must practically vanish for any beyond very low altitudes.

For the purpose then of ascertaining the limiting value for any possible lunar atmosphere, it will be requisite to determine within what limits there may be a horizontal refraction. Taking everything into consideration, the process best adapted to detect this horizontal refraction would be the different values given by a telescopic and occultation determination of the semi-diameter of the Moon. After having determined to within what value the horizontal refraction is limited by the known values of the lunar semi-diameters determined by these means, it would be possible to ascertain if any other phenomena still further limits it.

The Astronomer Royal has shown (*M. N.* xxv. p. 261, *Greenwich Observations*, 1864, Appendix I.,) as the result of the computations by Mr. Breen of the Moon's occultation semi-diameter, from observations of 295 occultations, that the following corrections are given by it to the Greenwich telescopic semi-diameter:—

By disappearance of stars at the dark limb	=	$-2''\cdot00$
By reappearance of stars at the dark limb	=	$-2\ \cdot40$

The similar observations at the bright limb give greater values for the semi-diameter, as might *a priori* be expected, both from the extreme delicacy and difficulty of the observations, and from the irradiation of the limb being apt to extinguish the light of the star before it reaches the actual limb. As Sir George Airy remarks: "We cannot be sensibly in error in saying that the Moon's occultation semi-diameter is less than the Moon's telescopic semi-diameter by $2''\cdot0$."

This excess is generally attributed to the results of irradiation at the bright limb increasing the telescopic semi-diameter, as the Greenwich value for this, referred to above, is considered a very exact determination; and although this may be the correct explanation, there can likewise be no doubt but that a portion, if not the whole, may possibly arise from a lunar atmosphere. Hansen's value for the semi-diameter of the Moon is regarded as being an exact determination of the true diameter, and is considerably less than the Greenwich telescopic semi-diameter; a reduction, however, of the occultation of stars for the years 1861 to 1870, gives a correction for the most favourable and accurate class of phenomena of $-1''\cdot70$ to this value, and the greater portion of this correction it appears cannot arise from the effect of irradiation. As a certain amount of irradiation must occur, and can but affect the telescopic semi-diameter, we cannot assume the whole correction to the telescopic semi-diameter as being due to the possible existence of a lunar atmosphere; and it would appear more satisfactory to allow for this, and say that these results show that an atmosphere may exist on the Moon capable of exerting a horizontal refraction of about one second of arc. Apart from other evidence, the observa-

tions of solar eclipses; especially those of 1860 and 1870 at Greenwich, where we have the effect of irradiation at its maximum, and exerted in diminishing the apparent lunar diameter; show that it is impracticable to ascribe to irradiation the excess of observed and computed semi-diameters over the occultation semi-diameter, as they unite in giving a maximum correction to Hansen's semi-diameter from the effects of irradiation of $-0''\cdot5$.

It remains now to determine on the basis of a possible horizontal refraction of one second of arc, the probable condition of the supposed lunar atmosphere. For the purpose of computing the horizontal refraction it will only be necessary to ascertain a probable law of decrease of density, as it has been shown very different rates of decrease give sensibly the same horizontal refraction, and considering the smallness of the horizontal refraction to be determined, it will be evident that considerable errors on the Earth would be insensible upon the Moon. As the rate of decrease of density depends upon the rate of decrease of temperature, and as there is no possibility of determining this, it will be necessary to assume some hypothesis. Reasoning from analogy, and taking into consideration the much slower decrease in density from the action of gravity, and the greater mobility from the decrease in density, it would appear we must have upon the Moon, not only a more equable decrease in density, but a greater uniformity in temperature, and consequently a slower decrease. It will be assumed that near the surface the decrease of temperature for increasing heights will be nearly equable, and slowly decreasing to a definite temperature at the upper limit of the atmosphere, an hypothesis similar to that of Laplace for the Earth. (*Mécanique Céleste*, tome x. ch. 1, § 7.) It may here be remarked that the more rapid the decrease of temperature the less the horizontal refraction becomes.

For the purpose of determining the refraction, it will be assumed that the atmosphere is unlimited in height, and that as before stated the temperature decreases slowly to a fixed minimum as the altitude increases. Unlike the Earth, the Moon suffers great variations in the temperature of its surface. Lord Rosse's researches show the temperature of the surface of the Moon to be very variable, reaching probably a maximum temperature of nearly two hundred degrees centigrade, and falling probably considerably below zero. This last is readily seen, considering the long lunar night, and the slight retarding influence exerted by a rare atmosphere free from moisture. As the temperature of space must be constant, it is evident this difference in the surface temperature will materially alter the constitution of the atmosphere and the amount of the refraction. We cannot from Lord Rosse's figures deduce any law for determining the variation of temperature, but it may approximately be taken as varying as the sine of the solar altitude. Consequently the maximum surface temperature of the following limb, would be at about the eighth day of the Moon's age, and the minimum about two days before full, while for the

other limb the similar periods would be the day after the third quarter and two days before full.

For the temperature of space, and therefore of the upper portion of the atmosphere, we have according to Fourier and Hopkins, about -50° centigrade, which value is adopted; but it is evident that by lowering this value we have not only a less horizontal refraction, but a greater mass of atmosphere, and therefore even more favourable conditions than those assumed. It is true by taking the temperature near -266° centigrade it renders the determination of the height of the atmosphere easier; but we have no reasons for assuming this to be in any sense probable; and Ivory has shown the height of the atmosphere does not sensibly alter the refraction. Upon, then, this value for the temperature of space we have for the minimum surface temperature of the Moon -30° cent. We must refer to one or two points before proceeding further. In considering the conditions of the atmosphere, the first mile from the surface is excluded, as liable to great local variations, and as not taking part in the refraction. This last is apparent, as for one hundred miles in width at the limb, the numerous mountains undergo no sensible diminution in height, and so must form a projecting ridge of quite this height. To determine the height of the atmosphere, as Laplace has shown that the height is of necessity limited, we must take the following consideration; for although to determine the refraction we may take the heights as unlimited, for other purposes we must determine its probable limit. We have physical reasons for believing that after the decrease of density has reached a certain amount, the elasticity of the atmosphere is destroyed at a medium low temperature, and this we must suppose reached at the limit of our own atmosphere. By taking then the density here as the minimum density, we can fix approximately the height of the lunar atmosphere as the point where this density is reached. Various heights for our atmosphere have been fixed, at from twenty-five to eighty miles, and by taking one hundred miles, it may be regarded as the extreme. Calculating upon Laplace's theory the density of the atmosphere at this height, we have, taking the surface density as unity, a number whose common logarithm is $\overline{12}\cdot3$. As the Moon's atmosphere will be about one five-hundredths of the density of that of the Earth at the surface; we may take the Moon's atmosphere as ceasing when its density is but the number whose logarithm is $\bar{9}\cdot0$ of the surface density. It is evident, however, that variation in this will exert very slight effect upon the density at the surface.

The physical conditions of an atmosphere upon the hypotheses adopted have been investigated by Laplace, Ivory, and Plana; but that of Ivory has been selected as not only generally preferred, but as from its simplicity admitting of being readily modified so as to suit conditions found on the Moon. Ivory's Memoirs are contained in the *Philosophical Transactions* for 1823 and 1838, and simple integrations of his equations are given by

Brünnow (*Spherical Astronomy*), and Bruhns (*Astronomische Strahlenbrechung*, pp. 148–159); they can also be readily deduced from Laplace's equations (*Mécanique Céleste*, tome x. ch. 1, § 7), and to one of these reference must be made for the demonstration of the equations used.

The notation used is as follows:—

a = radius of Moon.

x = altitude above surface of any point P.

δ, p, t = density, pressure, and temperature at P, and δ_0, p_0, t_0 ditto at surface.

l_0 = height of column of air of density δ_0, that at temperature t_0 would exert under the action of the Moon's gravity, a pressure equal to p_0.

$\alpha = 0{\cdot}000294\delta_0'$, putting this last in terms of the density of air at the surface of the Earth, and at 0° cent. and 760 millimetres pressure.

$\epsilon = 0{\cdot}003665$, its usual value as determined by Regnault.

Remembering that the height is to be considered unlimited, and as upon the hypotheses taken where m varies as the height of the atmosphere,

$$\frac{\delta}{\delta_0} = \left(1 - \frac{u}{m+1}\right)^m$$

we have for m infinite.

$$\frac{\delta}{\delta_0} = e^{-u},$$

and

$$\frac{p}{p_0} = e^{-u}\left(1 - f + fe^{-u}\right), \text{ and } \frac{p\,\delta_0}{p_0\,\delta} = 1 - f\left(1 - e^{-u}\right) = \frac{1+\epsilon t}{1+\epsilon t_0}$$

where

$$x = l_0\left\{(1-f)\,u + 2f\left(1 - e^{-u}\right)\right\}$$

where f is the function on the value given to which depends the rate at which the temperature decreases. It is evident from the third of these that as f increases the rate of decrease of temperature decreases, and that for $f = 0$ we have a uniform temperature, and that it cannot exceed unity.

Finally, for the horizontal refraction we have, putting r for the refraction and θ for zenith distance,

$$d\,r = \alpha(1+\alpha)\sin\theta\left\{\frac{d\left(1 - \frac{\delta}{\delta_0}\right)}{\sqrt{\left\{\cos^2\theta + 2\frac{x}{a} - 2\alpha + 2\alpha\frac{\delta}{\delta_0}\right\}}}\right\}$$

which equation can be expanded into terms of the well-known form $e^{T^2}\int_T^{\infty} e^{-tt} d\,t$, which can be readily integrated and reduced to the approximate form as under, with only insensible errors.

$$r = \alpha(1+\alpha)\ \frac{\sqrt{\pi\frac{a}{l_0}}}{\sqrt{2}}\left\{1 - \left(f - a\,\frac{a}{l_0}\right)\left(\sqrt{2}\cdot 1\right) + f\left(\frac{3}{2} - \sqrt{2}\right)\right\}$$

The term f can readily be deduced from the fact that it must be such as to make the temperature equal to that of space at the upper limit of the atmosphere, and not affect the surface temperature; and as from one of the equations above, we have $f = 1 - \frac{1 + \varepsilon\, t}{1 + \varepsilon\, t_0}$ approximately, by putting for t and t_0, the temperatures of the upper and lower limit of the atmosphere respectively, f is at once determined. It is also apparent that for the upper limit of the atmosphere, taking it as fixed by the value for the density before fixed, u will be approximately twenty, or the height of the atmosphere, for all surface temperatures remain sensibly constant, at a height slightly above half the Moon's radius.

Now, determining the surface density, so that the horizontal refraction at 25° centigrade will not exceed one second of arc, and be a convenient number for calculation: this will be found best represented by 0·0025 of the Earth's surface density at its standard pressure and temperature, or 760 millimetres and 0° centigrade. Putting then these values in the formulæ for the horizontal refraction, we have it amount to the following for the surface temperature stated :—

Surface temperature.		Horizontal refraction.	
−30° centigrade	...	1″·27	Mean temperature of Dark Limb.
0° ,,	...	1·03	
25° ,,	...	0·88	
100° ,,	...	0·59	Mean temperature of Bright Limb.
200° ,,	...	0·39	

It will be seen that these results are confirmatory of the cause of the difference between the occultation and telescopic semi-diameters, having their origin in the presence of a lunar atmosphere.

We have taken with Sir John Herschel the actual retardation at the limb, at its generally received value, namely, the horizontal refraction; and not twice this, as has been done. It can readily be shown that the retardation of an occultation is less than twice the horizontal refraction, as follows :—Although the angle between

the paths of a ray before and after refraction is evidently twice the horizontal refraction, yet it is apparent that the ray that would reach the observer is not that which would reach him in the absence of an atmosphere, but one lying much further from the limb; while that which would otherwise reach the observer after grazing the limb in the absence of a lunar atmosphere, would, if it were present, strike the surface of the Moon, and not reach the observer.

It remains now to notice the objections that have been raised to the existence of a lunar atmosphere, and it is evident that with one or two exceptions, as they were all directed against an atmosphere usually as dense, or even denser than our own, they are valueless as directed against one only one four-hundredths of this density. The phenomenon referred to by Mr. Proctor, in his Work on the Moon, as preventing the occultation of a star, could only arise from a lunar atmosphere much greater than our own, even were it not prevented from the rays from the Moon after refraction being divergent and not convergent, as he assumes in his illustration. It will also be apparent that for the density of the supposed atmosphere, no distortion of a star before occultation could possibly occur, and the same applies to the occultation of a planet such as *Jupiter* or *Saturn*, the maximum effect would be to increase the size of the planet by about one-thousandth; but in no case distort it. Dr. Huggins' observation (*M. N.* vol. xxv. p. 60) is evidently by no means delicate enough to detect the very slight effect capable of being exerted by an atmosphere of the density supposed. The effect of a lunar atmosphere upon an eclipse of the Sun, would, if of the density assumed, be sensibly the same as a diminution of the semi-diameter by about one second, or would be lost in the effects of irradiation. Finally, it can hardly be seriously urged that it could materially interfere with the observation of the reversal of the dark lines in the solar spectrum, considering the smallness of the horizontal refraction, and the extremely minute amount of scattering of the solar rays the supposed atmosphere could effect. No known objection yet raised appears to limit a possible lunar atmosphere more than the difference between the occultation and telescopic semi-diameter.

The real dimensions of the atmosphere shown to be possible upon the Moon's surface, can be best shown by the fact that its total weight above one square mile is about four hundred thousand tons; and that it bears nearly one eighth of the proportion of the Moon's mass, as the Earth's atmosphere does to the Earth's mass. The consideration of these features, however, had better be deferred until the fact of the presence of an atmosphere has been demonstrated.

In conclusion it is to be observed that the purport of the present paper has been to show that it is *possible* that a lunar atmosphere *may* exist, and to define its probable condition; the task of showing that a lunar atmosphere *does* exist requires different and more certain evidence.